

# RELATIVE HOMOLOGY AND MAXIMAL $l$ -ORTHOGONAL MODULES

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## INTRODUCTION

Maximal  $l$ -orthogonal modules for artin algebras were introduced by Iyama in [10] and [11], and were used to define a natural setting for developing a ‘higher dimensional’ Auslander-Reiten theory. In the second of these papers Iyama conjectures that the endomorphism ring of any two maximal  $l$ -orthogonal modules,  $M_1$  and  $M_2$ , are derived equivalent. He proves the conjecture for  $l = 1$ , and for  $l > 1$  he gives some orthogonality condition on  $M_1$  and  $M_2$ , such that the  $\text{End}_\Lambda(M_2)^{\text{op}}\text{-End}_\Lambda(M_1)$ -bimodule  $\text{Hom}_\Lambda(M_2, M_1)$  is tilting, which implies that the rings  $\text{End}_\Lambda(M_2)$  and  $\text{End}_\Lambda(M_1)$  are derived equivalent (see [9]). The purpose of this paper is to characterize tilting modules of the form  $\text{Hom}_\Lambda(M_2, M_1)$  in terms of the relative theories induced by the  $\Lambda$ -modules  $M_1$  and  $M_2$ , thus getting a generalization of Iyama’s result. Relative homological algebra, which we use throughout this paper, was developed by M. Auslander and Ø. Solberg in a series of three papers [1], [2], [3] and was used to study the representation theory of artin algebras.

Iyama’s conjecture for maximal  $l$ -orthogonal modules is motivated by the connection between maximal  $l$ -orthogonal modules and non-commutative crepant resolutions, which was shown in [11]. In particular Iyama proved that if  $R$  is a complete regular local ring of dimension  $d \geq 2$  and  $\Lambda$  is an  $R$ -order which is not an isolated singularity, then a Cohen-Macaulay  $\Lambda$ -module  $M$ , which is a generator and cogenerator for  $\text{mod } \Lambda$ , gives a non-commutative crepant resolution if and only if  $M$  is maximal  $(d - 2)$ -orthogonal. Considering maximal  $l$ -orthogonal modules as analogs of modules giving crepant resolutions, Iyama’s conjecture for maximal  $l$ -orthogonal modules is the analog of the Bondal-Orlov-Van den Bergh conjecture for crepant resolutions (see [5], [13]).

Let us fix some notation that we will use in the rest of this paper. By  $\Lambda$  we will denote an artin algebra and by  $\text{mod } \Lambda$  the category of finitely generated left  $\Lambda$ -modules. If  $M$  is in  $\text{mod } \Lambda$ , we denote by  $\text{add } M$  the full subcategory of  $\text{mod } \Lambda$ , consisting of summands of direct sums of  $M$ . A *generator-cogenerator* for  $\text{mod } \Lambda$ , is a  $\Lambda$ -module  $M$

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such that  $\text{add } M$  contains all the indecomposable projective and the indecomposable injective  $\Lambda$ -modules. Maximal  $l$ -orthogonal modules have this property.

In the first section, we recall some concepts from relative homological algebra and we study further the relative theories induced by generator-cogenerators for  $\text{mod } \Lambda$ . In the second section we give Iyama's definition for maximal  $l$ -orthogonal modules and state his conjecture. We prove our main theorem which is, as we already mentioned, a characterization of tilting modules of the form  $\text{Hom}_\Lambda(M_2, M_1)$  and we apply it to maximal  $l$ -orthogonal modules. In this way we are able to give a condition on two maximal  $l$ -orthogonal modules, so that the conjecture is true. We prove that Iyama's orthogonality condition implies our condition and give an example which shows that our condition is actually weaker.

## 1. RELATIVE AND ABSOLUTE HOMOLOGY

The aim of this section is first to recall some basic definitions and results on relative homology, and second to look especially at the relative theory induced by a generator-cogenerator for  $\text{mod } \Lambda$ . In particular we relate the global dimension of the endomorphism ring of a generator-cogenerator  $M$  in  $\text{mod } \Lambda$ , with the relative - with respect to  $M$  - global dimension of  $\Lambda$  (it will be defined precisely later). Moreover we will compare the relative homology induced by such a module, with the ordinary absolute homology. For unexplained terminology and results, we refer to [1] and [2].

Let  $F$  be an additive sub-bifunctor of  $\text{Ext}_\Lambda^1(-, -): (\text{mod } \Lambda)^{\text{op}} \times \text{mod } \Lambda \rightarrow \text{Ab}$ , where  $\text{Ab}$  denotes the category of abelian groups. A short exact sequence  $(\eta): 0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ , in  $\text{mod } \Lambda$ , is called *F-exact* if  $\eta$  is in  $F(C, A)$ . A  $\Lambda$ -module  $P$  is called *F-projective* if for any *F-exact* sequence  $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ , the sequence  $\text{Hom}_\Lambda(P, B) \rightarrow \text{Hom}_\Lambda(P, C) \rightarrow 0$  is exact. Dually, a  $\Lambda$ -module  $I$  is called *F-injective* if for any *F-exact* sequence  $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$ , the sequence  $\text{Hom}_\Lambda(B, I) \rightarrow \text{Hom}_\Lambda(A, I) \rightarrow 0$  is exact. We denote by  $\mathcal{P}(F)$  the full subcategory of  $\text{mod } \Lambda$  consisting of all *F-projective*  $\Lambda$ -modules and by  $\mathcal{I}(F)$  the one consisting of all *F-injective*  $\Lambda$ -modules. Note that if  $\mathcal{P}(\Lambda)$  and  $\mathcal{I}(\Lambda)$  denote the subcategories of projective and injective modules in  $\text{mod } \Lambda$  respectively, we have that  $\mathcal{P}(\Lambda) \subseteq \mathcal{P}(F)$  and  $\mathcal{I}(\Lambda) \subseteq \mathcal{I}(F)$ .

In this paper we work with sub-bifunctors of  $\text{Ext}_\Lambda^1(-, -)$  of the following special form. Let  $M$  be in  $\text{mod } \Lambda$ . For each pair of  $\Lambda$ -modules  $A$  and  $C$  we define

$$F_M(C, A) = \{0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0 \mid \text{Hom}_\Lambda(M, B) \rightarrow \text{Hom}_\Lambda(M, C) \rightarrow 0 \text{ is exact}\}$$

and dually

$$F^M(C, A) = \{0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0 \mid \text{Hom}_\Lambda(B, M) \rightarrow \text{Hom}_\Lambda(A, M) \rightarrow 0 \text{ is exact}\}.$$

It was shown in [1], that the above assignments give additive sub-bifunctors of  $\text{Ext}_\Lambda^1(-, -)$ . It is straightforward that  $\mathcal{P}(F_M) = \mathcal{P}(\Lambda) \cup \text{add } M$  and  $\mathcal{I}(F^M) = \mathcal{I}(\Lambda) \cup \text{add } M$ . Moreover, it is known that for any short exact sequence  $0 \rightarrow A \rightarrow B \rightarrow C \rightarrow 0$  and any  $\Lambda$ -module  $M$ , the sequence  $\text{Hom}_\Lambda(M, B) \rightarrow \text{Hom}_\Lambda(M, C) \rightarrow 0$  is exact if and only if the sequence  $\text{Hom}_\Lambda(B, \text{DTr } M) \rightarrow \text{Hom}_\Lambda(A, \text{DTr } M) \rightarrow 0$  is exact. Using this fact it is easy to see that  $F_M = F^{\text{DTr } M}$ , so  $\mathcal{I}(F_M) = \mathcal{I}(\Lambda) \cup \text{add } \text{DTr } M$ , and  $F^M = F_{\text{TrD } M}$ , so  $\mathcal{P}(F^M) = \mathcal{P}(\Lambda) \cup \text{add } \text{TrD } M$ . Hence we have a complete picture of the  $F$ -projectives and  $F$ -injectives for sub-bifunctors of the above form.

Another nice property of a sub-bifunctor  $F = F_M$  of  $\text{Ext}_\Lambda^1(-, -)$  is that it *has enough projectives*, in the sense that for any  $\Lambda$ -module  $C$ , there exists an  $F$ -exact sequence  $0 \rightarrow K \rightarrow P \rightarrow C \rightarrow 0$ , with  $P$  in  $\mathcal{P}(F)$ . Note that the map  $P \rightarrow C$  is nothing but a right  $\mathcal{P}(F)$ -approximation of  $C$  which we know is an epimorphism since  $\mathcal{P}(\Lambda) \subseteq \mathcal{P}(F)$ . Dually we see that  $F = F_M$  *has enough injectives*.

Let  $F = F_M$  and  $C$  in  $\text{mod } \Lambda$ . An  $F$ -projective resolution of  $C$  is an exact sequence

$$\cdots \rightarrow P_l \xrightarrow{f_l} P_{l-1} \rightarrow \cdots \rightarrow P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} C \rightarrow 0$$

where  $P_i$  is in  $\mathcal{P}(F)$  and each short exact sequence  $0 \rightarrow \text{Im } f_{i+1} \rightarrow P_i \rightarrow \text{Im } f_i \rightarrow 0$  is  $F$ -exact. Note that such a sequence exists for any  $\Lambda$ -module since  $F = F_M$  has enough projectives. The sequence is called a *minimal  $F$ -projective resolution* if in addition each  $P_i \rightarrow \text{Im } f_i$  is a minimal map. The  $F$ -projective dimension of  $C$ , which we denote by  $\text{pd}_F C$ , is defined to be the smallest  $n$  such that there exists an  $F$ -projective resolution

$$0 \rightarrow P_n \rightarrow \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow C \rightarrow 0.$$

If such  $n$  does not exist we set  $\text{pd}_F C = \infty$ . Dually, we can define the notion of a (*minimal*)  $F$ -injective resolution and the  $F$ -injective dimension,  $\text{id}_F C$ , of  $C$ . Then, the  $F$ -global dimension of  $\Lambda$  is defined as:

$$\text{gldim}_F \Lambda = \sup\{\text{pd}_F C \mid C \in \text{mod } \Lambda\} (= \sup\{\text{id}_F C \mid C \in \text{mod } \Lambda\}).$$

In the special case where  $M$  is a generator-cogenerator for  $\text{mod } \Lambda$ , we have the following nice connection between the global dimension of the endomorphism ring of  $M$  and the relative global dimension of  $\Lambda$ .

**Proposition 1.1.** *Let  $M$  be a generator-cogenerator for  $\text{mod } \Lambda$ . Then, for any positive integer  $l$ , the following are equivalent:*

$$(a) \text{ gldim } \text{End}_\Lambda(M) \leq l + 2$$

- (b)  $\text{gldim}_{F^M} \Lambda \leq l$
- (c)  $\text{gldim}_{F^M} \Lambda \leq l$ .

*Proof.* (a) $\Rightarrow$ (b) Let  $N$  be a  $\Lambda$ -module and let

$$\cdots \rightarrow M_l \rightarrow M_{l-1} \xrightarrow{f_{l-1}} \cdots \rightarrow M_1 \rightarrow M_0 \rightarrow N \rightarrow 0$$

be a minimal  $F_M$ -projective resolution of  $N$ . Set  $K_{l-1} = \text{Ker } f_{l-1}$ . We will show that  $K_{l-1}$  is in  $\text{add } M$ . Applying the functor  $\text{Hom}_\Lambda(M, -)$  to the above sequence, we get the long exact sequence

$$\begin{aligned} \eta_1 : \cdots \rightarrow \text{Hom}_\Lambda(M, M_l) \rightarrow \text{Hom}_\Lambda(M, M_{l-1}) \xrightarrow{\text{Hom}(M, f_{l-1})} \cdots \rightarrow \\ \text{Hom}_\Lambda(M, M_1) \rightarrow \text{Hom}_\Lambda(M, M_0) \rightarrow \text{Hom}_\Lambda(M, N) \rightarrow 0. \end{aligned}$$

Note that  $\text{Ker } \text{Hom}(M, f_{l-1}) = \text{Hom}(M, K_{l-1})$ . Let also

$$0 \rightarrow N \rightarrow M^0 \xrightarrow{f} M^1 \rightarrow \cdots$$

be the beginning of an  $F^M$ -injective resolution of  $N$ . Applying the functor  $\text{Hom}_\Lambda(M, -)$  to this sequence, we get the long exact sequence

$$\eta_2 : 0 \rightarrow \text{Hom}_\Lambda(M, N) \rightarrow \text{Hom}_\Lambda(M, M^0) \xrightarrow{\text{Hom}_\Lambda(M, f)} \text{Hom}_\Lambda(M, M^1) \rightarrow X \rightarrow 0$$

where the  $\text{End}_\Lambda(M)^{\text{op}}$ -module  $X$  is the cokernel of the map  $\text{Hom}_\Lambda(M, f)$ . Combining now the sequences  $\eta_1$  and  $\eta_2$  we get the long exact sequence

$$\begin{aligned} \cdots \rightarrow \text{Hom}_\Lambda(M, M_l) \rightarrow \text{Hom}_\Lambda(M, M_{l-1}) \rightarrow \cdots \rightarrow \text{Hom}_\Lambda(M, M_1) \rightarrow \\ \text{Hom}_\Lambda(M, M_0) \rightarrow \text{Hom}_\Lambda(M, M^0) \rightarrow \text{Hom}_\Lambda(M, M^1) \rightarrow X \rightarrow 0 \end{aligned}$$

Since the  $\Lambda$ -modules  $M_i$  for  $i = 0, 1, \dots$  and  $M^j$  for  $j = 0, 1$  are in  $\text{add } M$ , the  $\text{End}_\Lambda(M)^{\text{op}}$ -modules  $\text{Hom}_\Lambda(M, M_i), i = 0, 1, \dots$  and  $\text{Hom}_\Lambda(M, M^j), j = 0, 1$  are projective. So the  $\text{End}_\Lambda(M)^{\text{op}}$ -module  $\text{Hom}_\Lambda(M, K_{l-1})$  is an  $(l+2)$ -th syzygy of the  $\text{End}_\Lambda(M)^{\text{op}}$ -module  $X$ . Since  $\text{gldim } \text{End}_\Lambda(M) \leq l+2$ , the module  $\text{Hom}_\Lambda(M, K_{l-1})$  has to be a projective  $\text{End}_\Lambda(M)^{\text{op}}$ -module, so  $K_{l-1}$  is in  $\text{add } M$ . Thus  $\text{gldim}_{F^M} \Lambda \leq l$ .

(b) $\Rightarrow$ (a) Let  $X$  be an  $\text{End}_\Lambda(M)^{\text{op}}$ -module and let

$$\begin{aligned} \cdots \rightarrow \text{Hom}_\Lambda(M, M_{l+2}) \xrightarrow{d_{l+2}} \text{Hom}_\Lambda(M, M_{l+1}) \rightarrow \cdots \rightarrow \text{Hom}_\Lambda(M, M_2) \xrightarrow{d_2} \\ \text{Hom}_\Lambda(M, M_1) \xrightarrow{d_1} \text{Hom}_\Lambda(M, M_0) \rightarrow X \rightarrow 0 \end{aligned}$$

be a projective resolution of  $X$ . Since  $M$  is a generator for  $\text{mod } \Lambda$ , the functor  $\text{Hom}_\Lambda(M, -)$  is full and faithful hence we have that  $\text{Ker } d_i = \text{Hom}_\Lambda(M, K_i)$  where  $K_i$  is the kernel of the morphism  $f_i : M_i \rightarrow M_{i-1}$ , with  $\text{Hom}_\Lambda(M, f) = d_i$ , for  $i > 0$ . We will show that  $\text{Ker } d_{l+1}$  is a projective  $\text{End}_\Lambda(M)^{\text{op}}$ -module. The sequence

$$\cdots \rightarrow M_{l+2} \xrightarrow{f_{l+2}} M_{l+1} \rightarrow \cdots \rightarrow M_2 \rightarrow K_1 \rightarrow 0$$

is an  $F_M$ -projective resolution of  $K_1$ . Since  $\text{gldim}_{F_M} \Lambda \leq l$ , we have that  $K_{l+1}$  is in  $\text{add } M$  and hence  $\text{Hom}_\Lambda(M, K_{l+1})$  is a projective  $\text{End}_\Lambda(M)^{\text{op}}$ -module. So  $\text{gldim } \text{End}_\Lambda(M) \leq l + 2$ .

Thus we have proved the equivalence of (a) and (b). The proof of the equivalence of (a) and (c) is symmetric.  $\square$

Note that the above result was basically already known and can be found, for example in [8], in a different form. But, besides the fact that using relative homology helps in having a simpler statement, there is also the advantage of getting extra information about coresolutions of  $\Lambda$ -modules in  $\text{add DTr } M$  and resolutions of  $\Lambda$ -modules in  $\text{add TrD } M$ , since  $F_M = F^{\text{DTr } M}$  and  $F^M = F_{\text{TrD } M}$ . In particular, for selfinjective algebras, we have the following consequence.

**Corollary 1.2.** *Let  $\Lambda$  be a selfinjective artin algebra and  $X$  be in  $\text{mod } \Lambda$ . Then, for any positive integer  $l$ , the following are equivalent:*

- (a)  $\text{gldim } \text{End}_\Lambda(\Lambda \oplus X) \leq l + 2$ ,
- (b)  $\text{gldim } \text{End}_\Lambda(\Lambda \oplus \text{DTr } X) \leq l + 2$ ,
- (c)  $\text{gldim } \text{End}_\Lambda(\Lambda \oplus \text{TrD } X) \leq l + 2$ .

**Remark.** A straightforward consequence of the above corollary is that if  $\Lambda$  is a selfinjective artin algebra, then  $\Lambda \oplus X$  is an Auslander generator (that is, a generator-cogenerator for  $\text{mod } \Lambda$  such that the global dimension of its endomorphism ring gives the representation dimension of  $\Lambda$ ) if and only if  $\Lambda \oplus \text{DTr } X$  is an Auslander generator.

Let  $A$  and  $C$  be in  $\text{mod } \Lambda$ . Knowing that for sub-bifunctors  $F = F_M$  of  $\text{Ext}_\Lambda^1(-, -)$  any  $\Lambda$ -module has an  $F$ -projective and an  $F$ -injective resolution, one can define the right derived functors  $\text{Ext}_F^i(C, -)$  and  $\text{Ext}_F^i(-, A)$  of  $\text{Hom}_\Lambda(C, -)$  and  $\text{Hom}_\Lambda(-, A)$  respectively, in the same way as in the case  $\mathcal{P}(F) = \mathcal{P}(\Lambda)$ . Moreover, it can be proved that  $\text{Ext}_F^i(C, -)(A)$  is then isomorphic to  $\text{Ext}_F^i(-, A)(C)$  and that for  $i = 1$  we have that  $\text{Ext}_F^1(C, A) = F(C, A)$ .

Although  $\text{Ext}_F^1(C, A)$  is a subgroup of  $\text{Ext}_\Lambda^1(C, A)$ , very little is known about how the  $\text{Ext}_\Lambda^i(C, A)$ -groups and the relative  $\text{Ext}_F^i(C, A)$ -groups are related for  $i > 1$ . In the next proposition we consider a case where these two coincide. This will help us in the next section to compare Iyama's condition on maximal orthogonal modules to our result. For  $X$  and  $Y$  in  $\text{mod } \Lambda$  we write  $X \perp_k Y$  if  $\text{Ext}_\Lambda^i(X, Y) = (0)$ , for  $0 < i \leq k$ . Abusing the notation, we will write  $X \perp_k Y$  even for  $k = 0$  but this will mean no condition on  $X$  and  $Y$ .

**Proposition 1.3.** *Let  $M_1$  and  $M_2$  be in  $\text{mod } \Lambda$  such that  $M_2$  is a generator for  $\text{mod } \Lambda$  and  $M_1$  is a cogenerator for  $\text{mod } \Lambda$ . The following are equivalent for a positive integer  $k$ :*

- (a)  $M_2 \perp_k M_1$ ,
- (b) For any  $C$  in  $\text{mod } \Lambda$ ,  $\text{Ext}_{F_{M_2}}^i(C, M_1) \simeq \text{Ext}_\Lambda^i(C, M_1)$ , for  $0 < i \leq k$

(c) For any  $D$  in  $\text{mod } \Lambda$ ,  $\text{Ext}_{F_{M_1}}^i(M_2, D) \simeq \text{Ext}_\Lambda^i(M_2, D)$ , for  $0 < i \leq k$ .

*Proof.* (a) $\Rightarrow$ (b) Let  $C$  be in  $\text{mod } \Lambda$  and let

$$\cdots \rightarrow P_l \xrightarrow{f_l} \cdots \rightarrow P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} C \rightarrow 0$$

be a minimal  $F_{M_2}$ -projective resolution of  $M_1$ . Set  $K_{-1} = M_1$  and  $K_i = \text{Ker } f_i$ , for  $i = 0, 1, 2, \dots$ . We first show that for all  $i = -1, 0, 1, 2, \dots$  we have

$$\text{Ext}_{F_{M_2}}^1(K_i, M_1) \simeq \text{Ext}_\Lambda^1(K_i, M_1).$$

Applying the functor  $\text{Hom}_\Lambda(-, M_1)$  to the short exact sequence

$$0 \rightarrow K_{i+1} \rightarrow P_{i+1} \rightarrow K_i \rightarrow 0$$

we get the long exact sequence

$$\text{Hom}_\Lambda(P_{i+1}, M_1) \rightarrow \text{Hom}_\Lambda(K_{i+1}, M_1) \rightarrow \text{Ext}_\Lambda^1(K_i, M_1) \rightarrow \text{Ext}_\Lambda^1(P_{i+1}, M_1).$$

But since  $M_2 \perp_k M_1$  and  $P_{i+1}$  is in  $\text{add } M_2$ , we have  $\text{Ext}_\Lambda^1(P_{i+1}, M_1) = (0)$ . So  $\text{Ext}_\Lambda^1(K_i, M_1)$  is the cokernel of the map  $\text{Hom}_\Lambda(K_{i+1}, M_1) \rightarrow \text{Hom}_\Lambda(P_{i+1}, M_1)$ . The short exact sequence

$$0 \rightarrow K_{i+1} \rightarrow P_{i+1} \rightarrow K_i \rightarrow 0$$

is  $F_{M_2}$ -exact so we also have the following long exact sequence

$$\text{Hom}_\Lambda(P_{i+1}, M_1) \rightarrow \text{Hom}_\Lambda(K_{i+1}, M_1) \rightarrow \text{Ext}_{F_{M_2}}^1(K_i, M_1) \rightarrow \text{Ext}_{F_{M_2}}^1(P_{i+1}, M_1)$$

and since  $\text{Ext}_{F_{M_2}}^1(P_{i+1}, M_1) = (0)$ ,  $\text{Ext}_{F_{M_2}}^1(K_i, M_1)$  is the cokernel of the map  $\text{Hom}_\Lambda(K_{i+1}, M_1) \rightarrow \text{Hom}_\Lambda(P_{i+1}, M_1)$ . Hence

$$\text{Ext}_{F_{M_2}}^1(K_i, M_1) \simeq \text{Ext}_\Lambda^1(K_i, M_1),$$

for all  $i = -1, 0, 1, 2, \dots$ . In particular we have that

$$\text{Ext}_{F_{M_2}}^1(C, M_1) \simeq \text{Ext}_\Lambda^1(C, M_1).$$

Next we show that for all  $i = -1, 0, 1, 2, \dots$  and  $2 \leq j \leq k$

$$\text{Ext}_\Lambda^{j-1}(K_{i+1}, M_1) \simeq \text{Ext}_\Lambda^j(K_i, M_1).$$

To do this, we apply the functor  $\text{Hom}_\Lambda(-, M_1)$  to the short exact sequence

$$0 \rightarrow K_{i+1} \rightarrow P_{i+1} \rightarrow K_i \rightarrow 0$$

and we get the long exact sequences

$$\text{Ext}_\Lambda^{j-1}(P_{i+1}, M_1) \rightarrow \text{Ext}_\Lambda^{j-1}(K_{i+1}, M_1) \rightarrow \text{Ext}_\Lambda^j(K_i, M_1) \rightarrow \text{Ext}_\Lambda^j(P_{i+1}, M_1).$$

But since  $M_2 \perp_k M_1$  and  $P_{i+1}$  is in  $\text{add } M_2$ , we have that  $\text{Ext}_\Lambda^{j-1}(P_{i+1}, M_1) = 0$  and  $\text{Ext}_\Lambda^j(P_{i+1}, M_1) = 0$  which implies that

$$\text{Ext}_\Lambda^{j-1}(K_{i+1}, M_1) \simeq \text{Ext}_\Lambda^j(K_i, M_1).$$

Now using the above we can see that for all  $2 \leq i \leq k$  we have

$$\text{Ext}_{F_{M_2}}^i(C, M_1) \simeq \text{Ext}_{F_{M_2}}^{i-1}(K_0, M_1) \simeq \cdots \simeq \text{Ext}_{F_{M_2}}^1(K_{i-2}, M_1) \simeq$$

$$\mathrm{Ext}_\Lambda^1(K_{i-2}, M_1) \simeq \mathrm{Ext}_\Lambda^2(K_{i-3}, M_1) \simeq \cdots \simeq \mathrm{Ext}_\Lambda^i(C, M_1)$$

which completes the proof.

(b) $\Rightarrow$ (a) Set  $C = M_2$ . Then we have that  $\mathrm{Ext}_{F_{M_2}}^i(M_2, M_1) = (0)$ , since  $M_2$  is  $F_{M_2}$ -projective, hence  $\mathrm{Ext}_\Lambda^i(C, M_1) = (0)$ , for  $0 < i \leq k$ .

The proof of (a) $\Leftrightarrow$ (c) is symmetric.  $\square$

## 2. COTILTING AND MAXIMAL ORTHOGONAL MODULES

In this section we state and prove the main theorem and give the connections with Iyama's result. But let us start by recalling Iyama's definition for maximal  $l$ -orthogonal  $\Lambda$ -modules. Set  $M^{\perp_k} = \{Y \in \mathrm{mod} \Lambda \mid M \perp_k Y\}$  and  ${}^{\perp_k}M = \{X \in \mathrm{mod} \Lambda \mid X \perp_k M\}$ .

**Definition 2.1.** A  $\Lambda$ -module  $M$  is called *maximal  $l$ -orthogonal* if

$$M^{\perp_l} = \mathrm{add} M = {}^{\perp_l}M.$$

The following conjecture was stated in [11].

**Conjecture 2.2** (O. Iyama). Let  $M_1$  and  $M_2$  be maximal  $l$ -orthogonal in  $\mathrm{mod} \Lambda$ . Then their endomorphism rings,  $\mathrm{End}_\Lambda(M_1)$  and  $\mathrm{End}_\Lambda(M_2)$ , are derived equivalent.

Before we continue, we give a characterization of maximal orthogonal modules that can be found in a more general setting in [11, Proposition 2.2.2]. For convenience, we restate it and prove it here in the language of relative homology. We call a  $\Lambda$ -module  *$k$ -selforthogonal* if  $M \perp_k M$  holds.

**Proposition 2.3.** *Let  $M$  be in  $\mathrm{mod} \Lambda$  and  $l$  a positive integer. The following are equivalent for any integer  $k$ , such that  $0 \leq k \leq l$ .*

- (a)  $M$  is maximal  $l$ -orthogonal in  $\mathrm{mod} \Lambda$ ,
- (b)  $M$  is a generator-cogenerator for  $\mathrm{mod} \Lambda$ ,  $M$  is  $l$ -selforthogonal and  $\mathrm{pd}_{F_M} X \leq l - k$  for any  $X$  in  ${}^{\perp_k}M$ ,
- (c)  $M$  is a generator-cogenerator for  $\mathrm{mod} \Lambda$ ,  $M$  is  $l$ -selforthogonal and  $\mathrm{id}_{F_M} Y \leq l - k$  for any  $Y$  in  $M^{\perp_k}$ .

*Proof.* (a) $\Rightarrow$ (b) Let

$$\cdots \rightarrow M_{l-k} \xrightarrow{f_{l-k}} M_{l-k-1} \rightarrow \cdots \rightarrow M_1 \xrightarrow{f_1} M_0 \xrightarrow{f_0} C \rightarrow 0$$

be a minimal  $F_M$ -projective resolution of  $X$ . Set  $K_{-1} = X$  and  $K_i = \mathrm{Ker} f_i$ , for  $i = 0, 1, 2, \dots$ . We want to show that  $K_{l-k-1}$  is in  $\mathrm{add} M$ . In order to do this, we show that  $\mathrm{Ext}_\Lambda^j(M, K_{l-k-1}) = (0)$ , for all  $j = 1, 2, \dots, l$ . For all  $i$ , applying the functor  $\mathrm{Hom}_\Lambda(M, -)$  to the short exact sequence

$$0 \rightarrow K_{i+1} \rightarrow M_{i+1} \rightarrow K_i \rightarrow 0$$

we get the long exact sequence

$$\mathrm{Hom}_\Lambda(M, M_{i+1}) \rightarrow \mathrm{Hom}_\Lambda(M, K_i) \rightarrow \mathrm{Ext}_\Lambda^1(M, K_{i+1}) \rightarrow \mathrm{Ext}_\Lambda^1(M, M_{i+1}).$$

Since  $M$  is  $l$ -selforthogonal with  $l \geq 1$  and  $M_{i+1}$  is in  $\text{add } M$ , we have that  $\text{Ext}_\Lambda^1(M, M_{i+1}) = (0)$ . Moreover, since the short exact sequence  $0 \rightarrow K_{i+1} \rightarrow M_{i+1} \rightarrow K_i \rightarrow 0$  is  $F_M$ -exact, the map  $\text{Hom}_\Lambda(M, M_{i+1}) \rightarrow \text{Hom}_\Lambda(M, K_i)$  is an epimorphism. Hence  $\text{Ext}_\Lambda^1(M, K_{i+1}) = (0)$ . From the same short exact sequence we also get the long exact sequences

$$\text{Ext}_\Lambda^{j-1}(M, M_{i+1}) \rightarrow \text{Ext}_\Lambda^{j-1}(M, K_i) \rightarrow \text{Ext}_\Lambda^j(M, K_{i+1}) \rightarrow \text{Ext}_\Lambda^j(M, M_{i+1}).$$

For  $2 \leq j \leq l$ , since  $M$  is  $l$ -selforthogonal and  $M_{i+1}$  is in  $\text{add } M$ , we have that  $\text{Ext}_\Lambda^{j-1}(M, M_{i+1}) = (0)$  and  $\text{Ext}_\Lambda^j(M, M_{i+1}) = (0)$ , which implies that

$$\text{Ext}_\Lambda^{j-1}(M, K_i) \simeq \text{Ext}_\Lambda^j(M, K_{i+1}), \quad 2 \leq j \leq l$$

Now, using the above, we can compute the groups  $\text{Ext}_\Lambda^j(M, K_{l-k-1})$  as follows:

for  $2 \leq j \leq l - k$  we have

$$\text{Ext}_\Lambda^j(M, K_{l-k-1}) \simeq \text{Ext}_\Lambda^1(M, K_{l-k-j}) = (0)$$

and for  $j = l - k + s, 1 \leq s \leq k$  we have

$$\text{Ext}_\Lambda^{l-k+s}(M, K_{l-k-1}) \simeq \text{Ext}_\Lambda^s(M, X) = (0).$$

So we have

$$\text{Ext}_\Lambda^j(M, K_{l-k-1}) \simeq \text{Ext}_\Lambda^1(M, K_{l-k-j}) = (0), \quad 2 \leq j \leq l$$

and since  $M$  is maximal  $l$ -orthogonal this implies that  $K_{l-k-1}$  is in  $\text{add } M$ . Hence  $\text{pd}_{F_M} X \leq l - k$ .

(b) $\Rightarrow$ (a) Let  $X$  be in  $\text{mod } \Lambda$  such that  $\text{Ext}_\Lambda^i(X, M) = (0)$ , for  $0 < i \leq l$ . We will show that  $X$  is then in  $\text{add } M$ . Note that  $k \leq l$ , so  $\text{Ext}_\Lambda^i(X, M) = (0)$ , for  $0 < i \leq k$  or equivalently  $X$  is in  ${}^{\perp_k} M$ , hence by assumption  $\text{pd}_{F_M} X \leq l - k$ . If  $k = 0$ , this will imply that  $X$  is  $F_M$ -projective, hence  $X$  is in  $\text{add } M$ . Assume that  $k > 0$  and let

$$0 \rightarrow M_{l-k} \xrightarrow{f_{l-k}} M_{l-k-1} \rightarrow \cdots \rightarrow M_1 \xrightarrow{f_1} M_0 \xrightarrow{f_0} X \rightarrow 0$$

be a minimal  $F_M$ -projective resolution of  $X$ . Set  $K_{-1} = X$ ,  $K_i = \text{Ker } f_i$ , for  $i = 0, 1, \dots, l - k - 2$  and  $K_{l-k-1} = M_{l-k}$ . Then, for any  $i$ , applying the functor  $\text{Hom}_\Lambda(X, -)$  to the short exact sequence

$$0 \rightarrow K_i \rightarrow M_i \rightarrow K_{i-1} \rightarrow 0$$

we get long exact sequences

$$\text{Ext}_\Lambda^j(X, M_i) \rightarrow \text{Ext}_\Lambda^j(X, K_{i-1}) \rightarrow \text{Ext}_\Lambda^{j+1}(X, K_i) \rightarrow \text{Ext}_\Lambda^{j+1}(X, M_i).$$

Since  $M$  is  $l$ -selforthogonal and  $M_i$  is in  $\text{add } M$  for all  $i$ , we have that the first and last term of the above sequence vanish for all  $j = 1, 2, \dots, l - 1$ .

So we have

$$\text{Ext}_\Lambda^j(X, K_{i-1}) \simeq \text{Ext}_\Lambda^{j+1}(X, K_i)$$

for all  $i$  and for  $j = 1, 2, \dots, l - 1$ . But then we have

$$\text{Ext}_\Lambda^1(X, K_0) \simeq \text{Ext}_\Lambda^2(X, K_1) \simeq \cdots \simeq \text{Ext}_\Lambda^{l-k}(X, K_{l-k-1}) = \text{Ext}_\Lambda^{l-k}(X, M_{l-k}) = (0).$$



This implies that the short exact sequence  $0 \rightarrow K_0 \rightarrow M_0 \rightarrow X \rightarrow 0$  splits and hence  $X$  is in  $\text{add } M$ . So  $M$  is maximal  $l$ -orthogonal.

The proof of (a) $\Leftrightarrow$ (c) is symmetric.  $\square$

Setting  $k = 0$  in the above proposition, and using Lemma 1.1, we have the following nice characterization of maximal  $l$ -orthogonal modules.

**Corollary 2.4.** *Let  $M$  be in  $\text{mod } \Lambda$  and  $l$  a positive integer. The following are equivalent:*

- (a)  $M$  is maximal  $l$ -orthogonal,
- (b)  $M$  is a generator-cogenerator for  $\text{mod } \Lambda$ ,  $M$  is  $l$ -selforthogonal and  $\text{gldim } \text{End}_\Lambda(M) \leq l + 2$ .

Before we state and prove our main theorem we recall some definitions. A  $\Lambda$ -module  $M$  is called *cotilting*, if it has the following properties: (1)  $\text{Ext}_\Lambda^i(M, M) = (0)$ , for  $i > 0$ , (2)  $\text{id}_\Lambda M < \infty$  and (3)  $\mathcal{I}(\Lambda) \subseteq \widehat{\text{add } M}$ . Similarly, if  $F = F_M$  is a sub-bifunctor of  $\text{Ext}_\Lambda^1(-, -)$ , a  $\Lambda$ -module  $M$  is called  *$F$ -cotilting* if: (1)  $\text{Ext}_F^i(M, M) = (0)$ , for  $i > 0$ , (2)  $\text{id}_F M < \infty$  and (3)  $\mathcal{I}(F) \subseteq \widehat{\text{add}_F M}$ , where  $\widehat{\text{add}_F M}$  denotes the full subcategory of  $\text{mod } \Lambda$  consisting of all modules that have a finite resolution in  $\text{add } M$  which is in addition  $F$ -exact. The notions of *tilting* and  *$F$ -tilting* modules are defined dually.

For an artin algebra  $\Lambda$  it is known that when  $\text{gldim } \Lambda < \infty$ , a  $\Lambda$ -module  $M$  is tilting if and only if  $M$  is cotilting. The proof is based on the one to one correspondences between equivalence classes of tilting or cotilting  $\Lambda$ -modules and certain subcategories of  $\text{mod } \Lambda$  (see for example [12, Theorem 2.1]). A relative version of these correspondences is proved in [2] and using these one can prove the following:

**Lemma 2.5.** *Let  $F$  be a sub-bifunctor of  $\text{Ext}_\Lambda^1(-, -)$  with enough projectives and  $M$  a  $\Lambda$ -module. Assume that  $\text{gldim}_F \Lambda < \infty$ . Then  $M$  is  $F$ -tilting if and only if  $M$  is  $F$ -cotilting.*

We are now in position to give the main theorem of this paper.

**Theorem 2.6.** *Let  $\Lambda$  be an artin algebra with  $M_1$  and  $M_2$  two generator-cogenerators for  $\text{mod } \Lambda$ . Suppose that there exists some positive integer  $l$  such that  $\text{gldim } \text{End}_\Lambda(M_i) \leq l + 2$  for  $i = 1, 2$ . The following are equivalent:*

- (a)  $\text{Ext}_{F^{M_1}}^i(M_2, M_2) = (0)$  and  $\text{Ext}_{F^{M_2}}^i(M_1, M_1) = (0)$  for  $0 < i \leq l$ ,
- (b)  $M_2$  is an  $F^{M_1}$ -cotilting module,
- (c)  $M_1$  is an  $F^{M_2}$ -cotilting module,
- (d)  $\text{Hom}_\Lambda(M_2, M_1)$  is a cotilting  $\text{End}_\Lambda(M_2)^{\text{op}}\text{-End}_\Lambda(M_1)$ -bimodule.

*Proof.* (a) $\Rightarrow$ (b) First, since  $\text{gldim } \text{End}_\Lambda(M_1) \leq l + 2$ , applying Lemma 1.1, we see that  $\text{id}_{F^{M_1}} M_2 \leq l$ . Also, by assumption, we have  $\text{Ext}_{F^{M_1}}^i(M_2, M_2) =$

(0) for  $0 < i \leq l$  and since  $\text{id}_{F^{M_1}} M_2 \leq l$  we get  $\text{Ext}_{F^{M_1}}^i(M_2, M_2) = (0)$  for  $i > 0$ . It remains to show that  $\mathcal{I}(F^{M_1})$  is contained in  $\widehat{\text{add}_{F^{M_1}} M_2}$ . To do this, we consider a minimal  $F_{M_2}$ -projective resolution of  $M_1$

$$\cdots \rightarrow P_l \rightarrow P_{l-1} \xrightarrow{f_{l-1}} \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow M_1 \rightarrow 0$$

and we set  $K_{l-1} = \text{Ker } f_{l-1}$ . Since  $\text{gldim } \text{End}_\Lambda(M_2) \leq l + 2$ , applying Lemma 1.1 we see that  $K_{l-1}$  is  $F_{M_2}$ -projective, so it is in  $\text{add } M_2$ . Applying the functor  $\text{Hom}_\Lambda(-, M_1)$  to this sequence we get a long exact sequence

$$0 \rightarrow \text{Hom}_\Lambda(M_1, M_1) \rightarrow \text{Hom}_\Lambda(P_0, M_1) \rightarrow \text{Hom}_\Lambda(P_1, M_1) \rightarrow \cdots \rightarrow \text{Hom}_\Lambda(P_{l-1}, M_1) \rightarrow \text{Hom}_\Lambda(K_{l-1}, M_1).$$

But, by assumption,  $\text{Ext}_{F_{M_2}}^i(M_1, M_1) = (0)$  for  $0 < i \leq l$ , so the above sequence is exact and the map  $\text{Hom}_\Lambda(P_{l-1}, M_1) \rightarrow \text{Hom}_\Lambda(K_{l-1}, M_1)$  is an epimorphism, which implies that the sequence

$$0 \rightarrow K_{l-1} \rightarrow P_{l-1} \xrightarrow{f_{l-1}} \cdots \rightarrow P_1 \rightarrow P_0 \rightarrow M_1 \rightarrow 0$$

is  $F^{M_1}$ -exact. Thus  $M_2$  is an  $F^{M_1}$ -cotilting module.

(b) $\Rightarrow$ (a) Since  $M_2$  is an  $F^{M_1}$ -cotilting module,  $\text{Ext}_{F^{M_1}}^i(M_2, M_2) = (0)$  for  $i > 0$ . Moreover,  $\mathcal{I}(F^{M_1})$  is contained in  $\widehat{\text{add}_{F^{M_1}} M_2}$ , so there exists an  $F^{M_1}$ -exact sequence

$$(\eta): 0 \rightarrow P_n \rightarrow P_{n-1} \xrightarrow{f_{n-1}} \cdots \rightarrow P_1 \xrightarrow{f_1} P_0 \xrightarrow{f_0} M_1 \rightarrow 0$$

with  $P_i$  in  $\text{add } M_2$ , for all  $i$ . We will show that  $(\eta)$  is also  $F_{M_2}$ -exact. We set  $K_i = \text{Ker } f_i$  for  $i = 1, \dots, n-2$ ,  $K_0 = M_1$  and  $K_{n-1} = P_n$ . Then, for any  $i$ , applying the functor  $\text{Hom}_\Lambda(M_2, -)$  to the short exact sequence

$$(\eta_i): 0 \rightarrow K_i \rightarrow P_i \rightarrow K_{i-1}$$

we get long exact sequences

$$\text{Ext}_{F^{M_1}}^j(M_2, P_i) \rightarrow \text{Ext}_{F^{M_1}}^j(M_2, K_{i-1}) \rightarrow \text{Ext}_{F^{M_1}}^{j+1}(M_2, K_i) \rightarrow \text{Ext}_{F^{M_1}}^{j+1}(M_2, P_i)$$

for  $j > 0$ . But since  $P_i$  is in  $\text{add } M_2$ , for all  $i$ , the first and the last term of these sequences are zero, hence the two middle terms are isomorphic. So we have

$$\begin{aligned} \text{Ext}_{F^{M_1}}^1(M_2, K_i) &\simeq \text{Ext}_{F^{M_1}}^2(M_2, K_{i+1}) \simeq \cdots \simeq \text{Ext}_{F^{M_1}}^{n-i}(M_2, K_{n-1}) = \\ &\text{Ext}_{F^{M_1}}^{n-i}(M_2, P_n) = (0). \end{aligned}$$

for  $i = 0, 1, \dots, n-1$ . But by applying the functor  $\text{Hom}_\Lambda(M_2, -)$  to the short exact sequences  $(\eta_i)$ , we also get the long exact sequences

$$\text{Hom}_\Lambda(M_2, P_i) \rightarrow \text{Hom}_\Lambda(M_2, K_{i-1}) \rightarrow \text{Ext}_{F^{M_1}}^1(M_2, K_i)$$

and since  $\text{Ext}_{F^{M_1}}^1(M_2, K_i) = (0)$  for all  $i$ , we have that the map  $\text{Hom}_\Lambda(M_2, P_i) \rightarrow \text{Hom}_\Lambda(M_2, K_{i-1})$  is an epimorphism for all  $i$ , which implies that  $(\eta)$  is

$F_{M_2}$ -exact. We now apply the functor  $\text{Hom}_\Lambda(-, M_1)$  to  $(\eta)$  and we get the complex

$$0 \rightarrow \text{Hom}_\Lambda(M_1, M_1) \rightarrow \text{Hom}_\Lambda(P_0, M_1) \rightarrow \text{Hom}_\Lambda(P_1, M_1) \rightarrow \cdots \rightarrow \text{Hom}_\Lambda(P_n, M_1) \rightarrow 0$$

Since  $(\eta)$  is  $F_{M_2}$ -exact, the  $i$ -th-homology of the above complex is  $\text{Ext}_{F_{M_2}}^i(M_1, M_1)$ . But since  $(\eta)$  is  $F^{M_1}$ -exact, the above complex is acyclic. Hence  $\text{Ext}_{F_{M_2}}^i(M_1, M_1) = (0)$ , for  $i > 0$ , which completes the proof.

(a) $\Leftrightarrow$ (c) The proof is symmetric to the proof of (a) $\Leftrightarrow$ (b)

(b) $\Rightarrow$ (d) Since  $\text{gldim End}_\Lambda(M_1) \leq l + 2$ , by Lemma 1.1, we get  $\text{gldim}_{F^{M_1}} \Lambda \leq l$  and then, by Lemma 2.5, we have that  $M_2$  is an  $F^{M_1}$ -tilting  $\Lambda$ -module. But then, since  $\mathcal{I}(F^{M_1}) = \text{add } M_1$ , the module  $\text{Hom}_\Lambda(M_2, M_1)$  is a cotilting  $\text{End}_\Lambda(M_2)^{\text{op}}$ -module, as shown in [2]. This is equivalent to  $\text{Hom}_\Lambda(M_2, M_1)$  being a cotilting  $\text{End}_\Lambda(M_2)^{\text{op}}\text{-End}_\Lambda(M_1)$ -bimodule.

(d) $\Rightarrow$ (a) We first show that  $\text{Ext}_{F^{M_1}}^i(M_2, M_2) = (0)$  for  $0 < i \leq l$ . Recall that by Lemma 1.1 we have that  $\text{gldim}_{F^{M_1}} \Lambda \leq l$  and let

$$0 \rightarrow M_2 \rightarrow I_0 \rightarrow I_1 \rightarrow \cdots \rightarrow I_l \rightarrow 0$$

be a minimal  $F^{M_1}$ -injective resolution of  $M_2$ . Then by applying the functor

$\text{Hom}_\Lambda(-, M_1)$  we get a minimal projective resolution of the  $\text{End}_\Lambda(M_1)$ -module  $\text{Hom}_\Lambda(M_2, M_1)$

$$0 \rightarrow \text{Hom}_\Lambda(I_l, M_1) \rightarrow \text{Hom}_\Lambda(I_{l-1}, M_1) \rightarrow \cdots \rightarrow \text{Hom}_\Lambda(I_1, M_1) \rightarrow \text{Hom}_\Lambda(I_0, M_1) \rightarrow \text{Hom}_\Lambda(M_2, M_1) \rightarrow 0$$

Applying the functor  $\text{Hom}_{\text{End}_\Lambda(M_1)}(-, \text{Hom}_\Lambda(M_2, M_1))$  to the last sequence we get the following commutative exact diagram where the notation has been simplified

$$\begin{array}{ccccccc} 0 \twoheadrightarrow & ((M_2, M_1), (M_2, M_1)) & \twoheadrightarrow & ((I_0, M_1), (M_2, M_1)) & \twoheadrightarrow & \cdots \twoheadrightarrow & ((I_l, M_1), (M_2, M_1)) \twoheadrightarrow 0 \\ & \downarrow \wr & & \downarrow \wr & & & \downarrow \wr \\ 0 \longrightarrow & (M_2, M_2) & \longrightarrow & (M_2, I_0) & \longrightarrow & \cdots \longrightarrow & (M_2, I_l) \longrightarrow 0 \end{array}$$

From the above diagram we see that

$$\text{Ext}_{F^{M_1}}^i(M_2, M_2) \simeq \text{Ext}_{\text{End}_\Lambda(M_1)}^i(\text{Hom}_\Lambda(M_2, M_1), \text{Hom}_\Lambda(M_2, M_1)) = (0),$$

for  $i > 0$ , since  $\text{Hom}_\Lambda(M_2, M_1)$  is a cotilting  $\text{End}_\Lambda(M_1)$ -module.

Symmetrically, starting with a minimal  $F_{M_2}$ -projective resolution of  $M_1$  and applying the functor  $\text{Hom}_\Lambda(M_2, -)$  we can show that

$$\text{Ext}_{F_{M_2}}^i(M_1, M_1) \simeq \text{Ext}_{\text{End}_\Lambda(M_2)^{\text{op}}}^i(\text{Hom}_\Lambda(M_2, M_1), \text{Hom}_\Lambda(M_2, M_1)) = (0)$$

for  $i > 0$ . Here we are using that  $\text{Hom}_\Lambda(M_2, M_1)$  is a cotilting  $\text{End}_\Lambda(M_2)^{\text{op}}$ -module.  $\square$

The following easy consequence of the above theorem generalizes the Theorem 5.3.2 in [11].

**Corollary 2.7.** *Let  $M_1$  and  $M_2$  be maximal  $l$ -orthogonal modules in  $\text{mod } \Lambda$  such that  $\text{Ext}_{F_{M_1}}^i(M_2, M_2) = (0)$  and  $\text{Ext}_{F_{M_2}}^i(M_1, M_1) = (0)$  for  $0 < i \leq l$ . Then their endomorphism rings,  $\text{End}_\Lambda(M_1)$  and  $\text{End}_\Lambda(M_2)$ , are derived equivalent.*

*Proof.* Let  $M_1$  and  $M_2$  be maximal  $l$ -orthogonal modules in  $\text{mod } \Lambda$  satisfying the assumption of the Corollary. By Proposition 2.4 and Proposition 1.1 we have that  $M_1$  and  $M_2$  are generator-cogenerators for  $\text{mod } \Lambda$  with  $\text{gldim } \text{End}_\Lambda(M_i) \leq l + 2$ ,  $i = 1, 2$ . Then using Theorem 2.6 and we get that  $\text{Hom}_\Lambda(M_2, M_1)$  is a cotilting  $\text{End}_\Lambda(M_2)^{\text{op}}$ - $\text{End}_\Lambda(M_1)$ -bimodule and hence  $\text{End}_\Lambda(M_1)$  and  $\text{End}_\Lambda(M_2)$  are derived equivalent by a result of Happel [9].  $\square$

Although it is obvious from our Theorem 2.6 that Iyama's orthogonality condition on two maximal  $l$ -orthogonal modules  $M_1$  and  $M_2$  implies the vanishing of  $\text{Ext}_{F_{M_1}}^i(M_2, M_2)$  and  $\text{Ext}_{F_{M_2}}^i(M_1, M_1)$  for  $0 < i \leq l$ , it is interesting to give a direct proof of this fact. This is done in the next proposition.

**Proposition 2.8.** *Let  $M_1$  and  $M_2$  be maximal  $l$ -orthogonal in  $\text{mod } \Lambda$ . Assume that there exists a positive integer  $k$ , such that  $k \leq l \leq 2k + 1$  and  $M_2 \perp_k M_1$ . Then*

- (a)  $\text{Ext}_{F_{M_2}}^i(M_1, M_1) = (0)$ ,  $0 < i \leq l$ ,
- (b)  $\text{Ext}_{F_{M_1}}^i(M_2, M_2) = (0)$ ,  $0 < i \leq l$ .

*Proof.* (a) For  $0 < i \leq k$ , since  $M_2 \perp_k M_1$ , by Proposition 1.3 we have that

$$\text{Ext}_{F_{M_2}}^i(M_1, M_1) = \text{Ext}_\Lambda^i(M_1, M_1).$$

But  $M_1$  is  $l$ -orthogonal, so  $\text{Ext}_\Lambda^i(M_1, M_1) = (0)$ , for  $0 < i \leq k$ . Hence  $\text{Ext}_{F_{M_2}}^i(M_1, M_1) = (0)$ , for  $0 < i \leq k$ .

For  $i > k + 1$ , since  $l \leq 2k + 1$ , we have that  $i > l - k$ . But since  $M_2$  is maximal  $l$ -orthogonal, by Lemma 2.3 we have that  $\text{pd}_{F_{M_2}} M_1 \leq l - k$ . So  $\text{Ext}_{F_{M_2}}^i(M_1, M_1) = (0)$ , for  $i > k + 1$ .

For  $i = k + 1$ , if  $l < 2k + 1$ , we have again that  $i > l - k$ , so using the same argument as before we get that  $\text{Ext}_{F_{M_2}}^i(M_1, M_1) = (0)$ . It remains to show that  $\text{Ext}_{F_{M_2}}^{k+1}(M_1, M_1) = (0)$ , in the case where  $l = 2k + 1$ . In order to do this, consider a minimal  $F_{M_2}$ -projective resolution of  $M_1$

$$\cdots \rightarrow P_k \xrightarrow{f_k} P_{k-1} \rightarrow \cdots \rightarrow P_1 \xrightarrow{f_1} P_0 \rightarrow M_1 \rightarrow 0$$

and set  $K_i = \text{Ker } f_i$ , for  $i = 0, 1, \dots$ . Then we have

$$\begin{aligned} \text{Ext}_{F_{M_2}}^{k+1}(M_1, M_1) &\simeq \text{Ext}_{F_{M_2}}^k(K_0, M_1) \simeq \dots \simeq \text{Ext}_{F_{M_2}}^1(K_{k-1}, M_1) \simeq \\ &\text{Ext}_{\Lambda}^1(K_{k-1}, M_1) \simeq \dots \simeq \text{Ext}_{\Lambda}^k(K_0, M_1). \end{aligned}$$

To show these isomorphisms we use the same arguments as in the proof of 1.3 and we omit here the details. Applying the functor  $\text{Hom}_{\Lambda}(-, M_1)$  to the short exact sequence

$$0 \rightarrow K_0 \rightarrow P_0 \rightarrow M_1 \rightarrow 0$$

we get the long exact sequence

$$\text{Ext}_{\Lambda}^k(P_0, M_1) \rightarrow \text{Ext}_{\Lambda}^k(K_0, M_1) \rightarrow \text{Ext}_{\Lambda}^{k+1}(M_1, M_1).$$

Since  $M_1$  is  $l$ -orthogonal and  $k+1 < l$ , we have that  $\text{Ext}_{\Lambda}^{k+1}(M_1, M_1) = (0)$  and since  $M_2 \perp_k M_1$  and  $P_0$  is in  $\text{add } M_2$ , we have that  $\text{Ext}_{\Lambda}^k(P_0, M_1) = (0)$ . So  $\text{Ext}_{\Lambda}^k(K_0, M_1) = (0)$  and hence  $\text{Ext}_{F_{M_2}}^{k+1}(M_1, M_1) = (0)$ , which completes the proof.

The proof of (b) is symmetric.  $\square$

The converse of Proposition 2.8 is not in general true, as the following example shows.

**Example 2.9.** Let  $Q$  be the quiver

$$\begin{array}{ccc} 1 & \xrightarrow{\alpha} & 2 \\ & \searrow \gamma & \swarrow \beta \\ & 3 & \end{array}$$

and  $KQ$  the path algebra of  $Q$  over some field  $K$ . Let also  $I$  be the ideal of  $KQ$  generated by all paths of length 5 and consider the factor algebra  $\Lambda = KQ/I$ . Set

$$M_1 = P_1 \oplus P_2 \oplus P_3 \oplus S_1 \oplus P_3/\mathfrak{r}^2 P_3$$

and

$$M_2 = P_1 \oplus P_2 \oplus P_3 \oplus S_1 \oplus P_1/\mathfrak{r}^2 P_1,$$

where  $P_i$  denotes the indecomposable projective  $\Lambda$ -module corresponding to vertex  $i$ , for  $i = 1, 2, 3$  and  $S_1$  denotes the simple module in vertex 1. It is easy to verify that  $M_1$  and  $M_2$  are maximal 2-orthogonal modules in  $\text{mod } \Lambda$ . Moreover, the modules  $M_1$  and  $M_2$  are connected via the following exact sequence:

$$(\eta): 0 \rightarrow P_1/\mathfrak{r}^2 P_1 \rightarrow S_1 \oplus P_1 \rightarrow S_1 \oplus P_3 \rightarrow P_3/\mathfrak{r}^2 P_3 \rightarrow 0.$$

Observe that the above sequence is both an  $F_{M_2}$ -exact and an  $F^{M_1}$ -exact sequence. Hence  $(\eta)$  can be viewed both as an  $F_{M_2}$ -projective resolution of  $P_3/\mathfrak{r}^2 P_3$  and as an  $F^{M_1}$ -injective resolution of  $P_1/\mathfrak{r}^2 P_1$ . But then, if we apply the functor  $\text{Hom}_{\Lambda}(-, M_1)$  to  $(\eta)$ , the resulting

complex is acyclic, since  $(\eta)$  is  $F^{M_1}$ -exact, so  $\text{Ext}_{F^{M_2}}^i(P_3/\mathfrak{r}^2 P_3, M_1) = (0)$ , for  $i > 0$  and hence  $\text{Ext}_{F^{M_2}}^i(M_1, M_1) = (0)$ , for  $i > 0$ . Also, if we apply the functor  $\text{Hom}_\Lambda(M_2, -)$  to  $(\eta)$ , the resulting complex is acyclic, since  $(\eta)$  is  $F^{M_2}$ -exact, so  $\text{Ext}_{F^{M_1}}^i(M_2, P_1/\mathfrak{r}^2 P_1) = (0)$ , for  $i > 0$  and hence  $\text{Ext}_{F^{M_1}}^i(M_2, M_2) = (0)$ , for  $i > 0$ . Thus, we have shown that  $M_1$  and  $M_2$  satisfy the conclusions (a) and (b) of Proposition 2.8. Next we consider the short exact sequence

$$0 \rightarrow P_3/\mathfrak{r}^2 P_3 \rightarrow P_1/\text{Soc } P_1 \rightarrow P_1/\mathfrak{r}^2 P_1 \rightarrow 0.$$

The sequence is non split, so  $\text{Ext}_\Lambda^1(P_1/\mathfrak{r}^2 P_1, P_3/\mathfrak{r}^2 P_3) \neq (0)$ , hence  $\text{Ext}_\Lambda^1(M_2, M_1) \neq (0)$  which implies that Iyama's orthogonality condition does not hold for  $M_1$  and  $M_2$ .

In fact, in the above example, we can compute all maximal 2-orthogonal  $\Lambda$ -modules and we can then see that they are all connected with sequences which have the properties of  $(\eta)$ , meaning that we can get one from the other by exchanging one indecomposable summand using approximations. Thus, for the above example, Iyama's conjecture is true. We complete this section with a proposition showing the connection between these exchange sequences and the vanishing of the relative  $\text{Ext}_F$ .

**Proposition 2.10.** *Let  $M_i = N \oplus X_i$  be in  $\text{mod } \Lambda$  such that  $N$  is a generator-cogenerator for  $\text{mod } \Lambda$  and  $X_i$  are indecomposable not contained in  $\text{add } N$ , for  $i = 1, 2$ . Assume that there exists a positive integer  $l$  such that  $\text{gldim } \text{End}_\Lambda(M_i) \leq l + 2$ , for  $i = 1, 2$ . Then the following are equivalent:*

- (a)  $\text{Ext}_{F^{M_1}}^i(M_2, M_2) = (0)$  and  $\text{Ext}_{F^{M_2}}^i(M_1, M_1) = (0)$  for  $0 < i \leq l$ ,
- (b) *there exists an exact sequence*

$$(\eta): 0 \rightarrow X_2 \rightarrow N_0 \xrightarrow{f_0} N_1 \rightarrow \cdots \rightarrow N_m \xrightarrow{f_m} X_1 \rightarrow 0$$

*where each map  $\text{Ker } f_j \rightarrow M_j$  is a minimal left add  $N$ -approximation, each map  $M_j \rightarrow \text{Im } f_j$  is a minimal right add  $N$ -approximation and  $(\eta)$  is in addition  $F^{M_1}$ -exact and  $F^{M_2}$ -exact.*

*Proof.* (a) $\Rightarrow$ (b) Using Theorem 2.6 we have that  $\text{Hom}_\Lambda(M_2, M_1)$  is a cotilting  $\text{End}_\Lambda(M_1)$ -module. Also we know that  $\text{Hom}_\Lambda(M_1, M_1)$  is trivially a cotilting  $\text{End}_\Lambda(M_1)$ -module. So the  $\text{End}_\Lambda(M_1)$ -modules  $\text{Hom}_\Lambda(X_2, M_1)$  and  $\text{Hom}_\Lambda(X_1, M_1)$  are complements of the almost complete cotilting  $\text{End}_\Lambda(M_1)$ -module  $\text{Hom}_\Lambda(N, M_1)$ . Then, by [7], there exists an exact sequence

$$0 \rightarrow \text{Hom}_\Lambda(X_1, M_1) \xrightarrow{f_m^*} \text{Hom}_\Lambda(N_m, M_1) \rightarrow \cdots \xrightarrow{f_1^*} \text{Hom}_\Lambda(N_1, M_1) \xrightarrow{f_0^*} \text{Hom}_\Lambda(N_0, M_1) \rightarrow \text{Hom}_\Lambda(X_2, M_1) \rightarrow 0$$

where each map  $\text{Im } f_j^* \rightarrow \text{Hom}_\Lambda(N_j, M_1)$  is a minimal left add  $\text{Hom}_\Lambda(N, M_1)$ -approximation and each  $\text{Hom}_\Lambda(N_j, M_1) \rightarrow \text{Coker } f_j^*$  is a minimal right add  $\text{Hom}_\Lambda(N, M_1)$ -approximation. Since  $M_1$  is a cogenerator for  $\text{mod } \Lambda$ , we have that for all  $j$ ,  $f_j^* = \text{Hom}_\Lambda(f_j, M_1)$  for some  $f_j: M_j \rightarrow M_{j+1}$  and the sequence

$$(\eta): 0 \rightarrow X_2 \rightarrow N_0 \xrightarrow{f_0} N_1 \xrightarrow{f_1} \cdots \rightarrow N_m \xrightarrow{f_m} X_1 \rightarrow 0$$

is  $F^{M_1}$ -exact. Moreover each map  $\text{Ker } f_j \rightarrow M_j$  is a minimal left add  $N$ -approximation, each map  $M_j \rightarrow \text{Im } f_j$  is a minimal right add  $N$ -approximation. It remains to show that  $(\eta)$  is in also  $F_{M_2}$ -exact. To do this, we apply the functor  $\text{Hom}_\Lambda(M_2, -)$  to  $(\eta)$  and we get the complex

$$0 \rightarrow \text{Hom}_\Lambda(M_2, X_2) \rightarrow \text{Hom}_\Lambda(M_2, N_0) \rightarrow \text{Hom}_\Lambda(M_2, N_1) \rightarrow \cdots \rightarrow \text{Hom}_\Lambda(M_2, N_m) \rightarrow \text{Hom}_\Lambda(M_2, X_1) \rightarrow 0.$$

But  $(\eta)$  can be viewed as an  $F^{M_1}$ -injective resolution of  $X_2$  and then the  $j$ -th-homology of the above complex is  $\text{Ext}_{F^{M_1}}^j(M_2, X_2)$  which is, by assumption, zero. So the complex is acyclic which implies that  $(\eta)$  is  $F_{M_2}$ -exact.

(b) $\Rightarrow$ (a) Assume that there exists a sequence  $(\eta)$  as in (b). Then  $(\eta)$  can be viewed both as an  $F^{M_1}$ -injective resolution of  $X_2$  and as an  $F_{M_2}$ -projective resolution of  $X_1$ . If we apply the functor  $\text{Hom}_\Lambda(M_2, -)$  to  $(\eta)$ , the resulting complex will be acyclic since  $(\eta)$  is  $F_{M_2}$ -exact, hence  $\text{Ext}_{F^{M_1}}^i(M_2, X_2) = (0)$ , for  $i > 0$ , which implies that  $\text{Ext}_\Lambda^i(M_2, M_2) = (0)$ , for  $i > 0$ , since  $N$  is  $F^{M_1}$ -injective. Similarly, if we apply the functor  $\text{Hom}_\Lambda(-, M_1)$  to  $(\eta)$ , the resulting complex will be acyclic since  $(\eta)$  is  $F^{M_1}$ -exact, hence  $\text{Ext}_{F_{M_2}}^i(X_1, M_1) = (0)$ , for  $i > 0$ , which implies that  $\text{Ext}_\Lambda^i(M_2, M_2) = (0)$ , for  $i > 0$ , since  $N$  is  $F_{M_2}$ -injective.  $\square$

Note that the integer  $m$  that appears in the sequence  $(\eta)$  of the above proposition can be at most  $l - 1$ , by Proposition 1.1.

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